Sc_n = Schmidt number for power law model, $K\cdot[(3n+1)/4n]^n(8U_m/de)^{n-1}/D\cdot\rho$
Sh = Sherwood number $k\cdot de/D$
U = velocity, cm/s
U_m = mean velocity, cm/s

Greek Letters

ρ = density, g/cm^3
β = velocity gradient at the wall, s^{-1}
$\Omega(a, n)$ = dimensionless flow rate
$\lambda(a, n)$ = dimensionless radius for maximum velocity, r/R
$\Gamma(a, n)$ = function defined in Equation (10), dimensionless
τ_{rz} = local shear stress, dyne/cm^2
$\phi(a)$ = function defined in Equation (3), dimensionless

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Comments on Sufficiency Conditions for Constrained Extrema

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This journal has devoted space (Law and Fariss, 1971; Reklaitis, 1972) to the analysis of equality constrained optimization problems of the type

Extremize $f(\mathbf{x})$

subject to,

$$g_j(\mathbf{x}) = 0, \quad j = 1, 2, \ldots, m \tag{1}$$

The solution usually consists of two parts. In part one, the critical points are located, while in part two, the critical points are classified. Classification involves identifying the critical points as corresponding to either a maximum, a minimum, or a saddle. In nonexceptional cases, the critical points may be located by using either constrained derivatives or Lagrange multipliers. The remainder of this note will be directed to some of the computational aspects involved in the classification of critical points. In particular, for both the constrained derivative and the Lagrange multiplier approaches, methods will be presented for the computation of $\mathbf{H}_c$, which is the constrained Hessian matrix. This matrix is useful since if $\mathbf{H}_c$ is negative (positive) definite, the critical point being tested will correspond to a maximum (minimum). Also, a method for implementing Hancock's test will be presented.

CONSTRAINED DERIVATIVES

The **G** matrix is of rank r. By proper choice of the indexes of the x_α, $\alpha = 1, 2, \ldots, n$, and the $g_j(\mathbf{x})$, $j = 1, 2, \ldots, m$, the leading $(r \times r)$ portion of **G** may be made nonsingular. This submatrix of **G** is called **J**, while the $(r \times n - r)$ matrix in the north-east corner of **G** is called **C**. The first r components of **x** are designated as state variables, while the remaining $(n - r)$ components of **x** are called decision variables. From the chain rule

$$\mathbf{J}\, d\mathbf{s} + \mathbf{C}\, d\mathbf{d} = 0 \tag{2}$$

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Since **J** has an inverse it follows that

$$d\mathbf{s} = -\mathbf{P}\, d\mathbf{d} \tag{3}$$

as well as

$$d\mathbf{x} = \begin{bmatrix} -\mathbf{P} \\ \mathbf{I} \end{bmatrix} (d\mathbf{d}) \tag{4}$$

Wilde and Beightler (1967) show that first-order constrained derivatives may be computed from

$$\frac{\delta(\cdot)}{\delta d_a} = \frac{\partial(\cdot)}{\partial d_a} - P_{ia}\frac{\partial(\cdot)}{\partial s_i} \tag{5}$$

where the summation convention has been used. Second-order constrained derivatives of f are given by

$$\frac{\delta^2 f}{\delta d_a \delta d_b} = \left[\frac{\partial}{\partial d_a} - P_{ia}\frac{\partial}{\partial s_i}\right]\left[\frac{\partial f}{\partial d_b} - P_{jb}\frac{\partial f}{\partial s_j}\right] \tag{6}$$

Further manipulation yields

$$\frac{\delta^2 f}{\delta d_a \delta d_b} = P_{ia}\frac{\partial^2 f}{\partial s_i \partial s_j}P_{jb} - P_{ia}\frac{\partial^2 f}{\partial s_i \partial d_b} - \frac{\partial^2 f}{\partial d_a \partial s_j}P_{jb} + \frac{\partial^2 f}{\partial d_a \partial d_b} - \left[\frac{\delta P_{jb}}{\delta d_a}\right]\left[\frac{\partial f}{\partial s_j}\right] \tag{7}$$

The matrix equivalent of Equation (7) is

$$\mathbf{H}_c = [-\mathbf{P}^T, \mathbf{I}]\,\mathbf{H}\begin{bmatrix} -\mathbf{P} \\ \mathbf{I} \end{bmatrix} + \mathbf{T} \tag{8}$$

where the first four terms on the right side of Equation (7) correspond to the first term on the right side of Equation (8), and

$$\mathbf{T} = -\left\|\left[\frac{\delta P_{jb}}{\delta d_a}\right]\left[\frac{\partial f}{\partial s_j}\right]\right\| \tag{9}$$

Owing to the occurrence of a triply indexed quantity in the definition of **T**, further matrix reduction of Equation (9) is impossible. It may be shown, however, that the b^{th}

column of $\mathbf{T}$ may be written as

$$\mathbf{t}_b = -[\nabla_c(\mathbf{p}_b^T)][(\nabla_s f)] \tag{10}$$

and that the a^{th} row of $\mathbf{T}$ is given by

$$\mathbf{t}^a = -(\nabla_s f)^T\left(\frac{\delta \mathbf{P}}{\delta d_a}\right) \tag{11}$$

Thus, $\mathbf{T}$ may be computed from Equations (9), (10), or (11), whichever appears most convenient. $\mathbf{H}_c$ may then be gained from Equation (8).

LAGRANGE MULTIPLIERS

In this method, one forms the function

$$F = f - \lambda_k g_k, \quad 1 \leq k \leq r \tag{12}$$

and then demands that all of the first-order derivatives of F with respect to the x_α, $\alpha = 1, 2, \ldots, n$, are zero. These equations along with Equation (1) are usually sufficient to determine the location of the critical points as well as the values of the λ_k.

Consider now the computation of $\delta g_k/\delta d_b$. From Equation (5), one gets

$$\frac{\delta g_k}{\delta d_b} = \frac{\partial g_k}{\partial d_b} - P_{ib}\frac{\partial g_k}{\partial s_i} \tag{13}$$

$$= C_{kb} - P_{ib}\, J_{ki} \tag{14}$$

From the definition of $\mathbf{P}$, it is clear that

$$J_{ki}\, P_{ib} = C_{kb} \tag{15}$$

Thus, from Equation (14)

$$\frac{\delta g_k}{\delta d_b} \equiv 0 \tag{16}$$

It immediately follows that

$$\frac{\delta^2(\lambda_k g_k)}{\delta d_a \delta d_b} \equiv 0 \tag{17}$$

Replacement of f in Equation (7) by $(\lambda_k g_k)$ gives

$$0 = P_{ia}\frac{\partial^2(\lambda_k g_k)}{\partial s_i \partial s_j} P_{jb} - P_{ia}\frac{\partial^2(\lambda_k g_k)}{\partial s_i \partial d_b} - \frac{\partial^2(\lambda_k g_k)}{\partial d_a \partial s_j} P_{jb} + \frac{\partial^2(\lambda_k g_k)}{\partial d_a \partial d_b} - \left[\frac{\partial P_{jb}}{\partial d_a}\right]\left[\frac{\partial(\lambda_k g_k)}{\partial s_j}\right] \tag{18}$$

Subtraction of Equation (18) from Equation (7) yields

$$\frac{\delta^2 f}{\delta d_a \delta d_b} = P_{ia}\frac{\partial^2 F}{\partial s_i \partial s_j} P_{bj} - P_{ia}\frac{\partial^2 F}{\partial s_i \partial d_b} - \frac{\partial^2 F}{\partial d_a \partial s_j} P_{jb} + \frac{\partial^2 F}{\partial d_a \partial d_b} - \left[\frac{\delta P_{jb}}{\delta d_a}\right]\left[\frac{\partial F}{\partial s_j}\right] \tag{19}$$

The last term of Equation (19) is zero, since $\partial F/\partial s_j$ is set equal to zero as a necessary condition for locating the critical points. The matrix form of Equation (19) is

$$\mathbf{H}_c = [-\mathbf{P}^T, \mathbf{I}]\, \mathbf{H}_F \begin{bmatrix} -\mathbf{P} \\ \mathbf{I} \end{bmatrix} \tag{20}$$

HANCOCK'S TEST

The quadratic form to be tested is

$$\phi = (d\,\mathbf{d})^T\, \mathbf{H}_c\, (d\,\mathbf{d}) \tag{21}$$

Incorporating Equations (20) and (4), we get

$$\phi = (d\mathbf{x})^T\, \mathbf{H}_F\, (d\mathbf{x}) \tag{22}$$

Not all of the components of $(d\mathbf{x})$ are capable of independent variation owing to Equation (2), which is now rewritten as

$$[\mathbf{J}, \mathbf{C}]\, (d\mathbf{x}) = \mathbf{0} \tag{23}$$

Hancock (1960) appends the additional normalizing constraint

$$(d\mathbf{x})^T(d\mathbf{x}) = 1 \tag{24}$$

and then from Equation (22) solves for the critical values of ϕ subject to the constraints given by Equations (23) and (24). These critical values of ϕ are called e values and are given implicitly by

$$\det\left[\begin{array}{c|c} \mathbf{H}_F - e\,\mathbf{I} & \begin{matrix}\mathbf{J}^T \\ \mathbf{C}^T\end{matrix} \\ \hline \mathbf{J}, \quad \mathbf{C} & \mathbf{0} \end{array}\right] = 0 \tag{25}$$

Once the e values for a given critical point become known, the nature of ϕ (and $\mathbf{H}_c$) also become known. If all the e values are positive (negative), then the $\mathbf{H}_c$ is positive (negative) definite, etc.

These same e values may be gained by finding the critical values of ϕ as given by Equation (21) when subject to Equation (24), which may be rewritten as

$$(d\mathbf{d})^T\, \mathbf{B}\, (d\,\mathbf{d}) = 1 \tag{26}$$

The e values may be shown to be given by

$$\det[\mathbf{B}^{-1}\mathbf{H}_c - e\,\mathbf{I}] = 0 \tag{27}$$

Thus, the e values are the eigenvalues of $\mathbf{B}^{-1}\mathbf{H}_c$. Since $\mathbf{B}$ and $\mathbf{B}^{-1}$ may be shown to be positive definite, there is no problem concerning the existence of $\mathbf{B}^{-1}$.

REMARKS ON EQUATION (25)

The e values are the eigenvalues of

$$\left[\begin{array}{c|c} \mathbf{H}_F & \begin{matrix}\mathbf{J}^T \\ \mathbf{C}^T\end{matrix} \\ \hline \mathbf{J}, \quad \mathbf{C} & e\mathbf{I} \end{array}\right]$$

There are myriad ways of finding eigenvalues, and when hand calculation is used most of them will work in this case. The unique point of the calculation centers around the matrix explicitly containing the eigenvalues which are sought.

For computer solution, the method of interpolation that is presented by Faddeeva (1959) appears to be the most straightforward. The left side of Equation (25) may be shown to be a polynomial of degree $(n - r)$. This polynomial may be found by evaluating the determinant at $(n - r + 1)$ distinct values of e followed by collocation of the data generated. Only the largest and smallest of the zeros of the polynomial need be known, and this may be accomplished in many ways. When the assumed values of e are taken to be $0, 1, 2, \ldots, (n - r)$, the collocation may be done with the aid of a difference table. Caution must be exercised, since this method is subject to possible error buildup.

Example 1

Let

$$p(e) = \det\begin{bmatrix} (13-e) & -5 & -2 & 1 \\ -5 & (13-e) & -2 & 2 \\ -2 & -2 & (10-e) & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix} = 0$$

be solved for its two e values. From the computed values of $p(0)$, $p(1)$, and $p(2)$, the following difference table may be computed:

e	$p(e)$	Δp	$\Delta^2 p$
0	−2 790		
		391	
1	−2 399		−28
		363	
2	−2 036		

Thus

$$p(e) = -2\,790 + \frac{391}{1!}e - \frac{28}{2!}e(e-1) = -2\,790 + 405e - 14e^2$$

The e values are 11.3 and 17.6.

As an accuracy check, the $(n-r)^{\text{th}}$-order differences in the difference table should be

$$(-1)^n (n-r)! \det [\mathbf{J}\mathbf{J}^T + \mathbf{C}\mathbf{C}^T]$$

The e values may also be found from the eigenvalues of

$$\mathbf{B}^{-1}\mathbf{H}_c = \left(\frac{1}{14}\right)\begin{bmatrix} 280 & 116 \\ -35 & 125 \end{bmatrix}$$

Example 2

For the problem

extremize $[xy + yz]$

subject to

$$g(x, y, z) = x^2 + 2y^2 + 3z^2 - 4 = 0$$

the critical points and Lagrange multipliers are given in the accompanying table.

x	y	z	λ	Nature of $\mathbf{H}_c$	Remark
1	0	−1	0	Indefinite	Saddle
−1	0	1	0	Indefinite	Saddle
$\sqrt{6}/2$	1	$\sqrt{6}/6$	$\sqrt{6}/6$	Negative definite	Maximum
$-\sqrt{6}/2$	1	$-\sqrt{6}/6$	$-\sqrt{6}/6$	Positive definite	Minimum
$\sqrt{6}/2$	−1	$\sqrt{6}/6$	$-\sqrt{6}/6$	Positive definite	Minimum
$-\sqrt{6}/2$	−1	$-\sqrt{6}/6$	$\sqrt{6}/6$	Negative definite	Maximum

Calculate $\mathbf{H}_c$ by Equation (8) and also Equation (20) and fill in last two columns of the above table.

Using Equation (8)

Taking x to be the state variable and y and z to be the decision variables, we get

$$\mathbf{G} = [2x | 4y,\ 6z]$$

$$\mathbf{P} = \left[2\frac{y}{x},\ 3\frac{z}{x}\right]$$

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$[-\mathbf{P}^T, \mathbf{I}]\,\mathbf{H}\begin{bmatrix} -\mathbf{P} \\ \mathbf{I} \end{bmatrix} = \begin{bmatrix} -4\left(\frac{y}{x}\right) & 1 - 3\left(\frac{z}{x}\right) \\ 1 - 3\left(\frac{z}{x}\right) & 0 \end{bmatrix}$$

$\mathbf{t}^1$ and $\mathbf{t}^2$ may be calculated from Equations (11) and (5):

$$\nabla_s f = \frac{\partial}{\partial x}(xy + yz) = y$$

$$\frac{\delta \mathbf{P}}{\delta y} = \left(\frac{\partial}{\partial y} - P_{11}\frac{\partial}{\partial x}\right)\mathbf{P}$$

$$= \frac{2}{x}\left[1 + 2\left(\frac{y}{x}\right)^2,\ 3\left(\frac{y}{x}\right)\left(\frac{z}{x}\right)\right]$$

$$\frac{\delta \mathbf{P}}{\delta z} = \left(\frac{\partial}{\partial z} - P_{12}\frac{\partial}{\partial x}\right)\mathbf{P}$$

$$= \frac{3}{x}\left[2\left(\frac{y}{x}\right)\left(\frac{z}{x}\right),\ 1 + 3\left(\frac{z}{x}\right)^2\right]$$

Hence

$$\mathbf{T} = -\begin{bmatrix} 2\left(\frac{y}{x}\right)\left[1 + 2\left(\frac{y}{x}\right)^2\right] & 6\left(\frac{y}{x}\right)^2\left(\frac{z}{x}\right) \\ 6\left(\frac{y}{x}\right)^2\left(\frac{z}{x}\right) & 3\left(\frac{y}{x}\right)\left[1 + 3\left(\frac{z}{x}\right)^2\right] \end{bmatrix}$$

Finally, from (8)

$$\mathbf{H}_c = \begin{bmatrix} -2\left(\frac{y}{x}\right)\left[3 + 2\left(\frac{y}{x}\right)^2\right] & 1 - 3\left(\frac{z}{x}\right)\left[1 + 2\left(\frac{y}{x}\right)^2\right] \\ 1 - 3\left(\frac{z}{x}\right)\left[1 + 2\left(\frac{y}{x}\right)^2\right] & -3\left(\frac{y}{x}\right)\left[1 + 3\left(\frac{z}{x}\right)^2\right] \end{bmatrix}$$

Analysis of this matrix allows the nature of $\mathbf{H}_c$ to be determined.

Using Equation (20)

$$\mathbf{H}_F = \mathbf{H} - \lambda\left|\left|\frac{\partial^2 g}{\partial x_\alpha \partial x_\beta}\right|\right| = \begin{bmatrix} -2\lambda & 1 & 0 \\ 1 & -4\lambda & 1 \\ 0 & 1 & -6\lambda \end{bmatrix}$$

From Equation (20)

$$\mathbf{H}_c = \begin{bmatrix} -4\left(\frac{y}{x}\right)\left[1 + 2\lambda\left(\frac{y}{x}\right)\right] - 4\lambda & 1 - 3\left(\frac{z}{x}\right)\left[1 + 4\lambda\left(\frac{y}{x}\right)\right] \\ 1 - 3\left(\frac{z}{x}\right)\left[1 + 4\lambda\left(\frac{y}{x}\right)\right] & -6\lambda\left[1 + 3\left(\frac{z}{x}\right)^2\right] \end{bmatrix}$$

This rendition of $\mathbf{H}_c$ is numerically the same as that already given. Analysis of *either* form of $\mathbf{H}_c$ allows the latter two columns of the table to be filled in.

NOTATION

$\mathbf{B}$ $= \mathbf{P}^T\mathbf{P} + \mathbf{I}$
$\mathbf{C}$ $= (r \times n - r)$ matrix, $(\partial g_i/\partial d_a)$, north-east portion of $\mathbf{G}$
$\mathbf{d}$ $= (n - r \times 1)$ vector of decision variables, (d_a)
e = solution to Equations (25) or (27)
f = objective function
F = Lagrange function
g = constraining function
$\mathbf{G}$ $= (m \times n)$ matrix of rank r, $(\partial g_l/\partial x_\alpha)$
$\mathbf{H}$ $= (n \times n)$ Hessian matrix $(\partial^2 f/\partial x_\alpha \partial x_\beta)$
$\mathbf{H}_c$ $= (n - r \times n - r)$ constrained Hessian matrix, $(\delta^2 f/\delta d_a \delta d_b)$
$\mathbf{H}_F$ $= (n \times n)$ matrix, $(\partial^2 F/\partial x_\alpha/\partial x_\beta)$
$\mathbf{J}$ $= (r \times r)$ matrix, leading portion of $\mathbf{G}$, $(\partial g_i/\partial s_j)$
m = number of constraints
n = number of independent variables
p = polynomial of degree $(n - r)$
$\mathbf{P}$ $= (r \times n - r)$ matrix, $\mathbf{J}^{-1}\mathbf{C}$
r = rank of $\mathbf{G}$ and $\mathbf{J}$
$\mathbf{s}$ $= (r \times 1)$ vector of state variables, (s_i)
$\mathbf{t}^a$ $= (r \times n - r)$ vector, a^{th} row of $\mathbf{T}$
$\mathbf{t}_b$ $= (n - r \times 1)$ vector, b^{th} column of $\mathbf{T}$
$\mathbf{T}$ $= (n - r \times n - r)$ matrix, defined by Equation (9)
$\mathbf{x}$ $= (n \times 1)$ vector of independent variables, (x_α)

Other Symbols

a, b = dummy indexes, $1 \leq a, b \leq (n - r)$
i, j, k = dummy indexes, $1 \leq i, j, k \leq r$
l = dummy index, $1 \leq l \leq m$
α, β = dummy indexes, $1 \leq \alpha, \beta \leq n$
λ = Lagrange multiplier
ϕ = quadratic form
∇_c $= (n - r \times 1)$ vector operator $(\delta/\delta d_a)$
∇_s $= (r \times 1)$ vector operator, $(\partial/\partial s_i)$

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On the Possibility of Stabilizing a Simple Negative Feedback Control System by Increasing Controller Gain on a PID Controller

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The purpose of this note is to show under what conditions a simple closed loop system may be stable at low values of K_c, unstable at moderate values, and be restabilized at higher values of K_c. Such an effect is possible with PID control but not with PI or PD control systems (Coughanowr and Koppel, 1965).

Consider the characteristic equation (using Laplace transforms) for a system with three first-order transfer functions and a PID controller:

$$\tau_1\tau_2\tau_3 S^3 + (\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3)S^2 + (\tau_1 + \tau_2 + \tau_3)S + 1 + K_c[1 + \tau_D S + 1/(\tau_i S)] = 0 \tag{1}$$

If we let $\gamma_1 = \tau_1\tau_2\tau_3$, $\gamma_2 = \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3$, and $\gamma_3 = \tau_1 + \tau_2 + \tau_3$, and multiply Equation (1) out, we obtain

$$\gamma_1\tau_i S^4 + \gamma_2\tau_i S^3 + (\gamma_3 + K_c\tau_D)\tau_i S^2 + (K_c + 1)\tau_i S + K_c = 0 \tag{2}$$

The required values of K_c, τ_D, and τ_i that will insure stable operation can be determined using the Routh array (Coughanowr and Koppel, 1965). Since all the parameters are positive, the conditions for stability can be written as

$$\frac{\gamma_2(\gamma_3 + K_c\tau_D)\tau_i^2 - \gamma_1(K_c + 1)\tau_i^2}{\gamma_2\tau_i} > 0 \tag{3}$$

and

$$(K_c + 1)\tau_i - \frac{K_c\gamma_2^2\tau_i^2}{[\gamma_2(\gamma_3 + K_c\tau_D) - \gamma_1(K_c + 1)]\tau_i^2} > 0 \tag{4}$$

Equation (3) can be rewritten to show the dependence of K_c on τ_D:

$$K_c > \frac{\gamma_1/\gamma_2 - \gamma_3}{\tau_D - \gamma_1/\gamma_2} \quad \text{if} \quad \tau_D > \gamma_1/\gamma_2 \tag{5a}$$

$$K_c < \frac{\gamma_1/\gamma_2 - \gamma_3}{\tau_D - \gamma_1/\gamma_2} \quad \text{if} \quad \tau_D < \gamma_1/\gamma_2 \tag{5b}$$

It can be easily shown that $\gamma_3 > \gamma_1\gamma_2$ for all positive τ_1, τ_2, τ_3 [that is, $(\tau_1 + \tau_2 + \tau_3)\cdot(\tau_1\tau_2 + \tau_1\tau_3 + \tau_3\tau_2) > \tau_1\tau_2\tau_3$]. Consequently, any positive value of K_c will satisfy Equation (3) [or (5*a*)] if $\tau_D > \gamma_1\gamma_2$. For $\tau_D < \gamma_1\gamma_2$, some high values of K_c will not satisfy Equation (3) [or (5*b*)] and will lead to instabilities.

For a system to be stable, it must simultaneously

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